INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 16, 87–104 (1993)

A PROBLEM OF HEAT AND MASS TRANSFER: PROOF OF THE EXISTENCE CONDITION BY A FINITE DIFFERENCE METHOD

M. GAULTIER

UFR de Mathématiques et Informatique, 351 Cours de la Libération, F-33405 Talence Cédex, France

AND

M. LEZAUN AND F. VADILLO

Departamento de Matemática Aplicada, Estadistica e Investigación Operativa, Facultad de Ciencias, Universidad del País Vasco, Apartado 644, Bilbao, Spain

SUMMARY

We solve by a finite difference method a system of simultaneous non-linear partial differential equations which modelizes the transfer of heat and mass when a fluid evaporates from the hot wall and condenses on the cold wall of an upright rectangular cavity. The need to verify a certain condition associating the physical parameters of the fluid for the existence of steady state solutions is proved.

KEY WORDS Heat and mass transfer Laminar flow Coupled boundary conditions Navier–Stokes equations

1. INTRODUCTION

In this paper we deal with the numerical solution of a model describing the steady state, laminar flow of an inhomogeneous fluid in a two-dimensional bounded domain. The model allows for simultaneous transfer of heat and mass when, in the presence of a large concentration of an inert and non-condensable gas, a fluid evaporates from a hot vertical wall and part of it condenses on a cold vertical wall of a rectangular cavity.

The enclosure is between vertical liquid evaporating and condensing films. The temperatures and concentrations are uniform over the two gas-liquid interfaces. All the other sides are insulators to both energy and mass transfer. The fluid inside is a mixture of vapour and non-condensable gas. This model is useful in practical systems such as partial pressure distillation and solar desalination plants.

From general conservation laws and behaviour laws for fluids[1] we obtain the equations of the model (Figure 1):[2]

momentum

x-component $$v\frac{\partial v}{\partial x}+w\frac{\partial w}{\partial y}=\frac{-1}{\rho_0}\frac{\partial p'}{\partial x}-[\alpha_0(T-T_0)+\beta_0(G-G_0)]g+\bar{\mu}_0\Delta v,$$

y-component $$v\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial y}=\frac{-1}{\rho_0}\frac{\partial p'}{\partial y}+\bar{\mu}_0\Delta w,$$

0271–2091/93/020087–18$14.00

Received April 1991
Revised July 1992

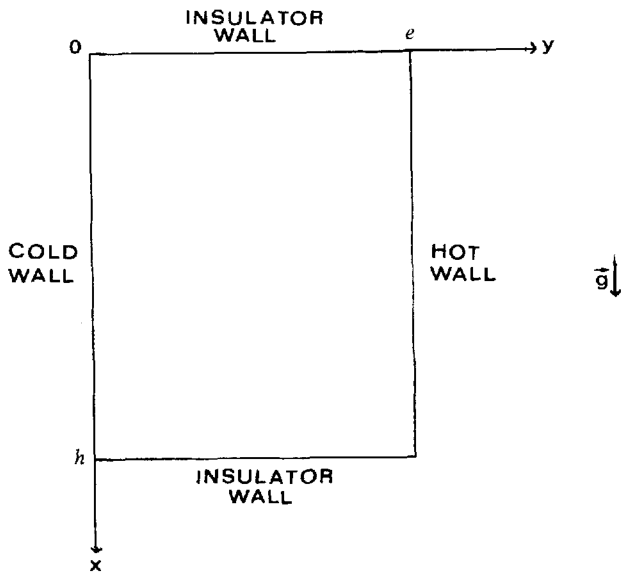

Figure 1

energy

$$v\frac{\partial T}{\partial x}+w\frac{\partial T}{\partial y}=\bar{v}_0\Delta T,$$

diffusion

$$v\frac{\partial G}{\partial x}+w\frac{\partial G}{\partial y}=D_{AB}\Delta G,$$

continuity

$$\frac{\partial v}{\partial x}+\frac{\partial w}{\partial y}=0,$$

where

e	cell width (cm)
h	cell height (cm)
v	x-direction velocity component ($cm\,s^{-1}$)
w	y-direction velocity component ($cm\,s^{-1}$)
T	temperature (°C or K)
T_0	average temperature of T_1 and T_2
G	mole fraction of phase-transferable component
G_0	average mole fraction of G_1 and G_2
g	gravitational acceleration ($cm\,s^{-2}$)
P	pressure ($dyn\,cm^{-2}$)
p'	combined pressure, $p-\rho_0 gx$ ($dyn\,cm^{-2}$)

ρ	density (gm cm^{-3})
α	thermal coefficient of volumetric expansion (°C^{-1} or K^{-1})
β	volumetric expansion coefficient (dimensionless)
$\bar{\mu}$	kinematic viscosity (cm^2 s^{-1})
$\bar{v}$	thermal diffusivity (cm^2 s^{-1})
D_{AB}	binary diffusion coefficient (cm^2 s^{-1}),

with subscripts

0	evaluated at T_0
1	gas–liquid interface on the condensing side
2	gas–liquid interface on the evaporating side.

The following implicit assumptions are made: (1) no chemical reaction; (2) hypothesis of Boussinesq; (3) no forced diffusion; (4) laminar and two-dimensional motion; (5) no thermal radiation.

The temperature is maintained constant over the two gas–liquid interfaces. Thus there is no reason to take into consideration latent heat of evaporation or condensation at vertical walls.

The boundary conditions are as follows:

cold wall

$$v=0, \qquad w=-b_1\frac{\partial G}{\partial y}, \qquad T=T_1, \qquad G=G_1,$$

hot wall

$$v=0, \qquad w=-b_2\frac{\partial G}{\partial y}, \qquad T=T_2, \qquad G=G_2,$$

insulator wall

$$v=0, \qquad w=-b\frac{\partial G}{\partial y}, \qquad \frac{\partial T}{\partial x}=\frac{\partial G}{\partial x}=0,$$

where b_i (cm^2 s^{-1}) is the mass transfer parameter at Γ_i ($i=1,2$).

The normal velocities at the interfaces are derived from mass flux balances. The tangential velocity at the evaporating interface is equal to zero. Since the condensation rate is a small percentage of the liquid feed rate, the tangential velocity at the condensing interface is set equal to zero. The slipping at insulated walls produced by the evaporation and condensation is modelized by $w=-b\ \partial G/\partial y$, where b is a function defined on $\Gamma_3\cup\Gamma_4$, with $b=b_1$ for $y=0$ and $b=b_2$ for $y=e$.

If non-dimensional co-ordinates, velocities, pressure, temperature and concentration are defined by

$$x_1=x/e, \qquad x_2=y/e, \qquad u_1=ve/b_1, \qquad u_2=we/b_1,$$
$$p=(d^2p')/(b_1^2\rho_0), \qquad \vartheta=(T-T_0)/(T_2-T_0), \qquad C=(G-G_0)/(G_2-G_0),$$

then the problem $(\mathscr{P})$ to solve is as follows.

Find (u, p, ϑ, C) satisfying the equations

$$
(\mathscr{P}) \begin{cases}
-\mu\Delta\mathbf{u} + \sum_{i=1}^{2} u_i\partial_i\mathbf{u} + \nabla p = -(\gamma_1\vartheta + \gamma_2 C, 0) \quad \text{in } \Omega, \\
\operatorname{div}(\mathbf{u}) = 0 \quad \text{in } \Omega, \\
-\nu_1\Delta\vartheta - \sum_{i=1}^{2} u_i\partial_i\vartheta = 0 \quad \text{in } \Omega, \\
-\nu_2\Delta C - \sum_{i=1}^{2} u_i\partial_i C = 0 \quad \text{in } \Omega \\
\text{and the boundary conditions} \\
u_1 = 0 \quad \text{and} \quad u_2 = -k\partial_2 C \quad \text{on } \Gamma, \\
\vartheta = C = -1 \quad \text{on } \Gamma_1, \\
\vartheta = C = 1 \quad \text{on } \Gamma_2, \\
\partial_1\vartheta = \partial_1 C = 0 \quad \text{on } \Gamma_3 \cup \Gamma_4,
\end{cases}
$$

where (Figure 2)

$$
\begin{aligned}
\Omega &= \{(x_1, x_2) \in \mathbb{R}^2;\, 0 < x_1 < L,\, 0 < x_2 < 1\}, \\
\Gamma_1 &= \{(x_1, 0) \in \mathbb{R}^2;\quad 0 < x_1 < L\}, \\
\Gamma_2 &= \{(x_1, 1) \in \mathbb{R}^2;\quad 0 < x_1 < L\}, \\
\Gamma_3 &= \{(0, x_2) \in \mathbb{R}^2;\quad 0 < x_2 < 1\}, \\
\Gamma_4 &= \{(L, x_2) \in \mathbb{R}^2;\quad 0 < x_2 < 1\}.
\end{aligned}
$$

$$
\mu = \bar{\mu}_0/b_1, \qquad \nu_1 = \bar{\nu}_0/b_1, \qquad \nu_2 = D_{AB}/b_1, \qquad \gamma_1 = \alpha_0 g e^3(T_2 - T_0)/b_1^2,
$$
$$
\gamma_2 = \beta_0 g e^3(G_2 - G_0)/b_1^2, \qquad k_1 = G_2 - G_0, \qquad k_2 = b_2(G_2 - G_0)/b_1
$$

and k is a function defined on $\Gamma_3 \cup \Gamma_4$, with $k = k_1$ for $x_2 = 0$ and $k = k_2$ for $x_2 = 1$.

We use the following notations:

$$
\partial_i v = \partial v/\partial x_i, \qquad \nabla v = (\partial_1 v, \partial_2 v), \qquad \mathbf{u} = (u_1, u_2), \qquad \Delta\mathbf{u} = (\Delta u_1, \Delta u_2), \qquad \operatorname{div}(\mathbf{u}) = \sum_{i=1}^{2} \partial_i u_i.
$$

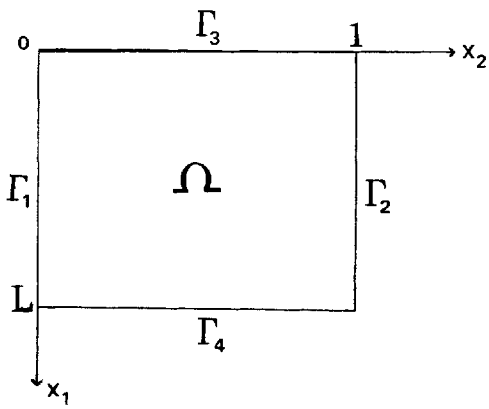

Figure 2

For the analytical resolution we suppose that the function k defined on $\Gamma_3 \cup \Gamma_4$ is the restriction of a function $k \in C^1(\bar{\Omega})$ with $k = k_i$ on $\Gamma_i (i = 1, 2)$. In the numerical experiments we have $k(x_1, x_2) = k_1 + (k_2 - k_1)x_2$.

Integrating $\operatorname{div}(\mathbf{u}) = 0$ over Ω, using Green's formula and the boundary conditions $u_1 = 0$ on $\Gamma_3 \cup \Gamma_4$ and $u_2 = -k_i \partial_2 C$ on Γ_i, we prove that the concentration C must satisfy the relation

$$k_1 \int_0^L \partial_2 C(x_1, 0)\, \mathrm{d}x_1 = k_2 \int_0^L \partial_2 C(x_1, 1)\, \mathrm{d}x_1.$$

Integrating the equation for C over Ω, using Green's formula, the boundary conditions and the above relation, we have

$$\frac{2k_1 k_2 - v_2(k_2 - k_1)}{v_2 + k_2} \int_0^L \partial_2 C(x_1, 0)\, \mathrm{d}x_1 = 0.$$

Thus the constants v_2, k_1 and k_2 must satisfy the equality

$$v_2 = \frac{2k_2 k_1}{k_2 - k_1}.$$

Let us assume that the preceding relation is satisfied.

The boundary conditions connecting the velocity and gradient of concentration exclude straightforward application of variational methods. In Section 2 we recall that under the condition

$$v > \lambda\rho + 2\sqrt{(\rho f)},$$

where

$$\frac{1}{v} = \operatorname{Max}\left(\frac{1}{\mu}, \frac{1}{v_1}, \frac{d}{v_2 - k_1} + \frac{d(k_1 + k_2)}{\sqrt{2}(v_2 - k_1)^2}\right), \quad \text{with } d = \sqrt{\left(1 + \frac{1}{\pi^2} + \frac{1}{\pi^4}\right)},$$

and $\rho(k, L, \gamma_1, \gamma_2)$, $\lambda(k, L, \gamma_1, \gamma_2)$ and $f(k, L, \gamma_1, \gamma_2)$ are positive constants depending on k, L, γ_1 and γ_2, problem $(\mathscr{P})$ has a unique solution.[3] This condition is satisfied if μ, v_1 and v_2 are quite large.

In Section 3 we look for the solution using an iterative method. Thus we have to solve four linear problems at each step.

In Section 4 we describe the formulation of a discrete approach to the four linear problems above. This formulation is compatible with the iterative method.

In Section 5 we give the numerical results for three representative cases.

Case 1. The sufficient existence and uniqueness condition is satisfied and the numerical scheme converges very quickly.

Case 2. The sufficient existence and uniqueness condition is not satisfied, but it is 'close', and the numerical scheme converges more slowly.

Case 3. The sufficient existence and uniqueness condition is largely not satisfied and no numerical solution of $(\mathscr{P})$ is found.

The numerical simulations obtained with the help of finite differences provide convincing evidence that a certain condition is necessary for the existence of a solution of $(\mathscr{P})$.

2. AN EXISTENCE AND UNIQUENESS THEOREM

To obtain homogeneous conditions in ϑ, C and $\mathbf{u}$ on Γ, we set $r(x_1, x_2)=2x_2-1$, $s(x_1, x_2)=-4x_2^3+6x_2^2$-1 and, for $\tilde{C}$ such that $v=0$ on $\Gamma_1\cup\Gamma_2$, $\partial_1 v=0$ on $\Gamma_3\cup\Gamma_4$ and $k_1\int_0^L \partial_2 v(x_1, 0)\,dx_1=k_2\int_0^L \partial_2 v(x_1, 1)\,dx_1$, $\mathbf{w}=(0, -k(\partial_2\tilde{C}+\partial_2 s))$. In problem $(\mathscr{P})$ we perform the substitution $\tilde{\vartheta}=\vartheta-r$, $\tilde{C}=C-s$, $\bar{\mathbf{u}}=\mathbf{u}-\beta(\mathbf{w})$ and $p=p+\mu q$, where $\beta(\mathbf{w})$ and q are the solution of the Stokes equations[4]

$$\begin{aligned} -\Delta(\beta(\mathbf{w}))+\nabla q&=0 \quad \text{in } \Omega,\\ \operatorname{div}(\beta(\mathbf{w}))&=0 \quad \text{in } \Omega,\\ \beta(\mathbf{w})&=\mathbf{w} \quad \text{on } \Gamma. \end{aligned}$$

Simple calculations lead us to the following problem:

$$-\mu\Delta\bar{\mathbf{u}}+\sum_{i=1}^{2}(\bar{\mathbf{u}}+\beta(\mathbf{w}))_i\partial_i(\bar{\mathbf{u}}+\beta(\mathbf{w}))+\nabla p'=-(\gamma_1(\tilde{\vartheta}+r)+\gamma_2(\tilde{C}+s), 0) \quad \text{in } \Omega, \tag{1}$$

$$\operatorname{div}(\bar{\mathbf{u}})=0 \quad \text{in } \Omega, \tag{2}$$

$$-v_1\Delta\tilde{\vartheta}+\sum_{i=1}^{2}(\bar{\mathbf{u}}+\beta(\mathbf{w}))_i\partial_i(\tilde{\vartheta}+r)=0 \qquad \text{in } \Omega, \tag{3}$$

$$-v_2\Delta\tilde{C}+\sum_{i=1}^{2}(\bar{\mathbf{u}}+\beta(\mathbf{w}))_i\partial_i(\tilde{C}+s)=v_2\Delta s \quad \text{in } \Omega, \tag{4}$$

$$\partial_1\tilde{\vartheta}=0 \quad \text{and} \quad \partial_1\tilde{C}=0 \quad \text{on } \Gamma_3\cup\Gamma_4, \tag{5}$$

$$\tilde{\vartheta}=\tilde{C}=0 \quad \text{on } \Gamma_1\cup\Gamma_2, \tag{6}$$

$$\bar{\mathbf{u}}=0 \quad \text{on } \Gamma. \tag{7}$$

We introduce the following spaces:

$$\begin{aligned} V&=\{\mathbf{u}\in(H_0^1(\Omega))^2;\ \operatorname{div}(\mathbf{u})=0 \text{ in } \Omega\},\\ Z&=\{v\in H^2(\Omega);\ v=0 \text{ on } \Gamma_1\cup\Gamma_2,\ \partial_1 v=0 \text{ on } \Gamma_3\cup\Gamma_4\},\\ Z_0&=\left\{v\in Z;\ k_1\int_0^L \partial_2 v(x_1, 0)\,dx_1=k_2\int_0^L \partial_2 v(x_1, 1)\,dx_1\right\},\\ X&=V\times Z\times Z_0. \end{aligned}$$

Let $\mathscr{A}=(\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3)$ and $\mathscr{F}=(\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3)$ be operators defined on X as follows: $\mathscr{A}_1$ is the isomorphism of V onto V^* defined by the V-elliptic continuous form

$$a(\bar{\mathbf{u}}, \bar{\mathbf{v}})=\mu\sum_{i,j=1}^{2}\int_\Omega \partial_i\bar{u}_j\partial_i\bar{v}_j\,dx \quad \forall\bar{\mathbf{u}}, \bar{\mathbf{v}}\in V;$$

$\mathscr{A}_2$ and $\mathscr{A}_3$ are the isomorphisms of Z onto $L^2(\Omega)$,

$$\begin{aligned} \mathscr{A}_2\tilde{\vartheta}&=-v_1\Delta\tilde{\vartheta} \quad \forall\tilde{\vartheta}\in Z,\\ \mathscr{A}_3\tilde{C}&=-v_2\Delta\tilde{C}+\operatorname{div}(\sigma\nabla\tilde{C}) \quad \forall\tilde{C}\in Z; \end{aligned}$$

and $\forall(\tilde{\mathbf{u}}, \tilde{\vartheta}, \tilde{C}) \in X$,

$$\mathscr{F}_1(\tilde{\mathbf{u}}, \tilde{\vartheta}, \tilde{C}) = -\sum_{i=1}^{2} (\tilde{\mathbf{u}}+\beta(\mathbf{w}))_i \partial_i(\tilde{\mathbf{u}}+\beta(\mathbf{w})) - (\gamma_1(\tilde{\vartheta}+r)+\gamma_2(\tilde{C}+s), 0),$$

$$\mathscr{F}_2(\tilde{\mathbf{u}}, \tilde{\vartheta}, \tilde{C}) = -\sum_{i=1}^{2} (\tilde{\mathbf{u}}+\beta(\mathbf{w}))_i \partial_i(\tilde{\vartheta}+r),$$

$$\mathscr{F}_3(\tilde{\mathbf{u}}, \tilde{\vartheta}, \tilde{C}) = -\sum_{i=1}^{2} (\tilde{\mathbf{u}}+\beta(\mathbf{w}))_i \partial_i(\tilde{C}+s) + \nu_2 \Delta s + \operatorname{div}(\sigma \nabla \tilde{C}).$$

Here $\sigma(x_1, x_2) = k_1 - (k_1+k_2)x_2$.

Remark 1

In Reference 3 it is proved that solving (1)–(7) is equivalent to solving in X the equation

$$\mathscr{A}(\tilde{\mathbf{u}}, \tilde{\vartheta}, \tilde{C}) = \mathscr{F}(\tilde{\mathbf{u}}, \tilde{\vartheta}, \tilde{C}). \tag{8}$$

To find the solutions of $\mathscr{A}u = \mathscr{F}u$ as fixed points of the operator $\mathscr{A}^{-1} \circ \mathscr{F}$, it is necessary that range $(\mathscr{F}) \subseteq \text{range}(\mathscr{A})$. Therefore we must add $\operatorname{div}(\sigma \nabla \tilde{C})$ to both sides of (4), where σ is a suitable function. Thus the initial problem is reduced to finding fixed points of the operator $\mathscr{A}^{-1} \circ \mathscr{F}$ on X.

Theorem 1.

If the inequality $\nu > \lambda\rho + 2\sqrt{(\rho \| \mathscr{F}(0) \| y)}$ is satisfied, where

$$\frac{1}{\nu} = \operatorname{Max}\left(\frac{1}{\mu}, \frac{1}{\nu_1}, \frac{d}{\nu_2 - k_1} + \frac{d(k_1+k_2)}{\sqrt{2}(\nu_2 - k_1)^2}\right), \quad \text{with } d = \sqrt{\left(1 + \frac{1}{\pi^2} + \frac{1}{\pi^4}\right)},$$

and $\rho(k, L, \gamma_1, \gamma_2)$ and $\lambda(k, L, \gamma_1, \gamma_2)$ are positive constants depending on k, L, γ_1 and γ_2, then there exists $R > 0$ such that in the ball of centre 0 and radius R the operator $T = \mathscr{A}^{-1} \circ \mathscr{F}$ has a unique fixed point.

Remark 2

The preceding condition is satisfied if μ, ν_1 and ν_2 are quite large. The calculation of the constants $\rho(k, L, \gamma_1, \gamma_2)$ and $\lambda(k, L, \gamma_1, \gamma_2)$ is very long and is given in Reference 3. For practical purposes we substitute $\| \mathscr{F}(0) \|_Y$ by an upper bound f computed in Reference 3. Therefore our actual condition for convergence,

$$\nu > \lambda\rho + 2\sqrt{(\rho f)}, \tag{9}$$

is more restrictive.

3. ITERATIVE METHOD

The solution of (8) is sought using an iterative method.

Letting $(\tilde{\mathbf{u}}^m, \tilde{\vartheta}^m, \tilde{C}^m) \in X$, we find $(\tilde{\mathbf{u}}^{m+1}, \tilde{\vartheta}^{m+1}, \tilde{C}^{m+1}) \in X$ such that

$$\mathscr{A}(\tilde{\mathbf{u}}^{m+1}, \tilde{\vartheta}^{m+1}, \tilde{C}^{m+1}) = \mathscr{F}(\tilde{\mathbf{u}}^m, \tilde{\vartheta}^m, \tilde{C}^m). \tag{10}$$

The limit $(\tilde{\mathbf{u}}, \tilde{\vartheta}, \tilde{C}) \in X$ of this sequence $(\tilde{\mathbf{u}}^m, \tilde{\vartheta}^m, \tilde{C}^m)$ is the solution of equation (8).

To find the solution of (10) given the definition of $\mathscr{A}$ and $\mathscr{F}$, the following four linear problems must be solved.

Problem ($\mathscr{P}_1$)

Find $(\beta(\mathbf{w}^m), q^m)\in(H^1(\Omega))^2\times L^2(\Omega)/\mathbb{R}$ such that

$$-\Delta(\beta(\mathbf{w}^m))+\nabla q^m=0 \quad \text{in } \Omega,$$
$$\operatorname{div}(\beta(\mathbf{w}^m))=0 \quad \text{in } \Omega,$$
$$\beta(\mathbf{w}^m)=\phi^m \quad \text{on } \Gamma,$$

where ϕ^m is the trace of $\mathbf{w}^m=(0,-k(\partial_2\tilde{C}^m+\partial_2 s))$ on Γ.

Problem ($\mathscr{P}_2$)

Find $(\bar{\mathbf{u}}^{m+1}, p'^{m+1})\in V\times L^2(\Omega)/\mathbb{R}$ such that

$$-\mu\Delta\bar{\mathbf{u}}^{m+1}+\nabla p'^{m+1}=-\sum_{i=1}^{2}(\bar{\mathbf{u}}^m+\beta(\mathbf{w}^m))_i\partial_i(\bar{\mathbf{u}}^m+\beta(\mathbf{w}^m))-(\gamma_1(\tilde{\vartheta}^m+r)+\gamma_2(\tilde{C}^m+s),0) \quad \text{in } \Omega,$$
$$\operatorname{div}(\bar{\mathbf{u}}^{m+1})=0 \quad \text{in } \Omega,$$
$$\bar{\mathbf{u}}^{m+1}=0 \quad \text{on } \Gamma.$$

Problem ($\mathscr{P}_3$)

Find $\tilde{\vartheta}^{m+1}\in H^2(\Omega)$ such that

$$-v_1\Delta\tilde{\vartheta}^{m+1}=-\sum_{i=1}^{2}(\bar{\mathbf{u}}^m+\beta(\mathbf{w}^m))_i\partial_i(\tilde{\vartheta}^m+r) \quad \text{in } \Omega,$$
$$\partial_1\tilde{\vartheta}^{m+1}=0 \quad \text{on } \Gamma_3\cup\Gamma_4, \qquad \tilde{\vartheta}^{m+1}=0 \quad \text{on } \Gamma_1\cup\Gamma_2.$$

Problem ($\mathscr{P}_4$)

Find $\tilde{C}^{m+1}\in H^2(\Omega)$ such that

$$-v_2\Delta\tilde{C}^{m+1}+\operatorname{div}(\sigma\nabla\tilde{C}^{m+1})=-\sum_{i=1}^{2}(\bar{\mathbf{u}}^m+\beta(\mathbf{w}^m))_i\partial_i(\tilde{C}^m+s)+v_2\Delta s+\operatorname{div}(\sigma\nabla\tilde{C}^m) \quad \text{in } \Omega,$$
$$\partial_1\tilde{C}^{m+1}=0 \quad \text{on } \Gamma_3\cup\Gamma_4, \qquad \tilde{C}^{m+1}=0 \quad \text{on } \Gamma_1\cup\Gamma_2.$$

Remark 3

The 'special move' used in the construction of operators $\mathscr{A}$ and $\mathscr{F}$ (see Remark 1) implies that $\tilde{C}^{m+1}$ satisfies

$$k_1\int_0^L \partial_2\tilde{C}^{m+1}(x_1,0)\,dx_1=k_2\int_0^L \partial_2\tilde{C}^{m+1}(x_1,1)\,dx_1.$$

Therefore $(\bar{\mathbf{u}}^{m+1}, \tilde{\vartheta}^{m+1}, \tilde{C}^{m+1})\in X$; the iterative method is therefore well-posed, since range $(\mathscr{F})\subseteq$ range $(\mathscr{A})$.

4. THE DISCRETE PROBLEM

In this section we describe a discrete approach for each of the four linear problems given in the last section.

4.1. Problems ($\mathscr{P}_1$) and ($\mathscr{P}_2$)

These problems are particular cases of the Stokes equations:

$$-\Delta\mathbf{u}+\nabla p=f \qquad \text{in } \Omega, \tag{11}$$

$$\operatorname{div}(\mathbf{u})=0 \qquad \text{in } \Omega, \tag{12}$$

$$\mathbf{u}=(0, b) \qquad \text{on } \Gamma, \tag{13}$$

where f and b are known and b satisfies

$$\int_0^L b(x_1, 0)\,\mathrm{d}x_1=\int_0^L b(x_1, 1)\,\mathrm{d}x_1$$

(in ($\mathscr{P}_1$) $f=0$, in ($\mathscr{P}_2$) $b=0$).

To solve (11)–(13), we use a discrete Galerkin approach in finite differences based on scheme II given in Reference 5. For simplicity we assume that Ω is the unit square and we consider meshes with uniform spacing.

On the unit square Ω we define the meshes (Figure 3)

$$\Omega_h=\{(ih, jh)\,|\,i=1, 2, \ldots, N-1; j=1, 2, \ldots, N-1\},$$

$$\Gamma_h=\{(ih, jh)\,|\,i=1, 2, \ldots, N-1, j=0; j=1, 2, \ldots, N-1, i=0\},$$

$$\partial\Omega_h=\{(ih, jh)\,|\,i=1, 2, \ldots, N \text{ when } j=0 \text{ and } j=N; j=1, 2, \ldots, N \text{ when } i=0 \text{ and } i=N\},$$

$$\tilde{\Omega}_h=\Omega_h\cup\Gamma_h.$$

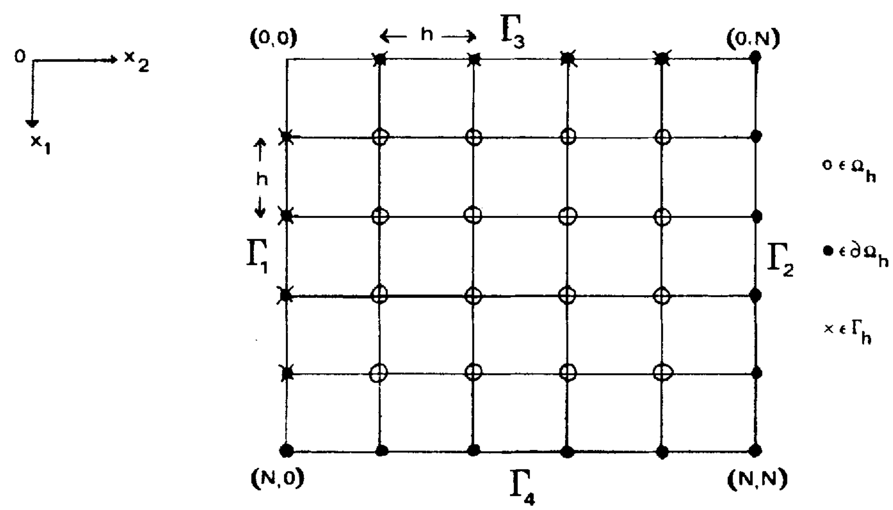

Figure 3

On these meshes we define mesh vectors $\mathbf{u}^h$ and mesh scalars ϕ^h. We also need to introduce the spaces

$$V_h = \{\mathbf{u}^h = (u_1^h, u_2^h) \text{ defined on } \Omega_h \cup \partial\Omega_h\},$$
$$V_h^0 = \{\mathbf{u}^h \in V_h \mid \mathbf{u}^h = 0 \text{ on } \partial\Omega_h\} \equiv \{\mathbf{u} \text{ defined on } \Omega_h\},$$
$$W_h = \{\phi^h \text{ scalar defined on } \tilde{\Omega}_h\}.$$

Finally, let $\mathscr{D}_h$ and $\mathscr{G}_h$ denote discrete finite difference operators approximating ∇ and div respectively. These operators $\mathscr{D}_h$: $V_h \to W_h$ and $\mathscr{G}_h$: $W_h \to V_h^0$ are defined respectively as

$$(\mathscr{D}_h \mathbf{u}^h)_{i,j} = (1/h)[(u_{1_{i+1,j}} - u_{1_{i,j}}) + \{u_{2_{i,j+1}} - u_{2_{i,j}}\}],$$

where $\mathbf{u}^h = (u_1, u_2) \in V_h$, and

$$(\mathscr{G}_h \phi^h)_{i,j} = (1/h)((\phi_{i,j} - \phi_{i-1,j}), (\phi_{i,j} - \phi_{i,j-1})).$$

On V_h^0 and W_h we consider the standard l^2 inner products

$$(\mathbf{u}^h, \mathbf{v}^h)_{V_h^0} = \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} u_{i,j} v_{i,j},$$

$$(\phi^h, \psi^h)_{W_h} = \sum_{\substack{i=0 \\ i+j \neq 0}}^{N-1} \sum_{j=0}^{N-1} \phi_{i,j} \psi_{i,j}.$$

Then it can be directly proved that $\mathscr{D}_h$ and $\mathscr{G}_h$ verify

$$(\mathscr{D}_h \mathbf{u}^h, \phi^h)_{W_h} = (\mathbf{u}^h, -\mathscr{G}_h \phi^h)_{V_h^0} \quad \forall \mathbf{u}^h \in V_h^0 \quad \text{and} \quad \forall \phi_h \in W_h. \tag{14}$$

Condition (14) implies that $\mathscr{D}_h$ and $-\mathscr{G}_h$ are adjoint and thus[6] we have the following decomposition of V_h^0.

Theorem 2

$V_h^0 = D_h \oplus G_h$, where

$$D_h = \{\mathbf{u}^h \in V_h^0 \mid \mathscr{D}_h \mathbf{u}^h = 0\},$$
$$G_h = \{\mathbf{u}^h \in V_h^0 \mid \exists \phi^h \in W_h \text{ such that } \mathbf{u}^h = \mathscr{G}_h \phi^h\}.$$

For this scheme we have dim $D_h = (N-2)^2$ and a basis for D_h is given by

$$\Phi_{i,j}^{l+i/2, m+1/2} = \begin{cases} (0, 1), & i = 1, \quad j = m+1, \\ (-1, 0), & i = l+1, \ j = m, \\ (1, -1), & i = l+1, \ j = m+1, \\ (0, 0), & \text{all other } i, j, \end{cases}$$

$$l = 1, 2, \ldots, N-2, \qquad m = 1, 2, \ldots, N-2.$$

Let the operator $\mathscr{N}_h$: $V_h \to V_h^0$ be defined as

$$(\mathscr{N}_h \mathbf{u}^h)_{i,j} = (-1/h^2)((\delta_1^h \mathbf{u}^h)_{i,j} + (\delta_2^h \mathbf{u}^h)_{i,j}),$$

where

$$\delta_1^h(\cdot)_{i,j}=(\cdot)_{i+1,j}-2(\cdot)_{i,j}+(\cdot)_{i-1,j},$$
$$\delta_2^h(\cdot)_{i,j}=(\cdot)_{i,j+1}-2(\cdot)_{i,j}+(\cdot)_{i,j-1}.$$

The difference equations approximating (11)–(13) are given by

$$\mathcal{N}_h\mathbf{u}^h=-\mathcal{G}_hp^h+f^h \quad \text{in } \Omega_h, \tag{15}$$

$$\mathcal{D}_h\mathbf{u}^h=0 \quad \text{in } \tilde{\Omega}_h, \tag{16}$$

$$\mathbf{u}^h=(0, b^h) \quad \text{on } \partial\Omega_h, \tag{17}$$

where f^h and b^h are appropriate discretizations of the data f and b respectively.

The decomposition of V_h^0 leads directly to a discrete Galerkin approximation. Indeed, let $\{\Phi_1, \Phi_2, \ldots, \Phi_m\}$, with $m=(N-1)^2$, be a basis for D_h and let $\mathbf{a}^h\in V_h$ satisfy $\mathcal{D}_h\mathbf{a}^h=0$ on $\tilde{\Omega}_h$ with $\mathbf{a}^h=(0, b^h)$ on $\partial\Omega_h$. Then the discrete Galerkin approximation $\mathbf{w}^h=\sum_{i=1}^m \alpha_i\Phi_i$ is defined as the solution of

$$(\mathcal{N}_h(\mathbf{w}^b+\mathbf{a}^h), \Phi_i)_{V_h^0}=(f^h, \Phi_i)_{V_h^0}, \quad i=1, 2, \ldots, m. \tag{18}$$

System (18) represents m (scalar) equations for the m (scalar) coefficients α_i and is equivalent to the finite difference system (15)–(17) in the following sense.

Theorem 3

If $\mathbf{u}^h$ and p^h satisfy (15)–(17), then $\mathbf{w}^h=\mathbf{u}^h-\mathbf{a}^h$ satisfies (18). Conversely, if $\mathbf{w}^h$ satisfies (18), then there exists a $p^h\in W_h$ such that $\mathbf{w}^h+\mathbf{a}^h$ and p^h satisfy (15)–(17).

To solve equation (18), one must construct the mesh vector $\mathbf{a}^h$. We have the following.

Lemma 1

A necessary and sufficient condition for the existence of the mesh vector $\mathbf{a}^h\in V_h$ satisfying $\mathcal{D}_h\mathbf{a}^h=0$ in $\tilde{\Omega}_h$ and $\mathbf{a}^h=(0, b^h)$ on $\partial\Omega_h$ is

$$\sum_{i=0}^{N-1} b_{i,N}-\sum_{i=1}^{N-1} b_{i,0}-b_{0,1}=0. \tag{19}$$

This condition arises from the summation of $\mathcal{D}_h\mathbf{a}^h=0$ over all nodes of $\tilde{\Omega}_h$.

4.2. Problem ($\mathcal{P}_3$)

In this problem we find $\tilde{\vartheta}$ satisfying

$$-\Delta\tilde{\vartheta}=g \quad \text{in } \Omega,$$
$$\partial_1\tilde{\vartheta}=0 \quad \text{on } \Gamma_3\cup\Gamma_4, \qquad \tilde{\vartheta}=0 \quad \text{on } \Gamma_1\cup\Gamma_2.$$

We approach this elliptic linear problem using a convergent standard method.[7] Thus we have the following finite approximation:

$$(-1/h^2)((\delta_1^h\tilde{\vartheta}^h)_{i,j}+(\delta_2^h\tilde{\vartheta}^h)_{i,j})=g_{i,j}, \quad 1\leqslant i,j\leqslant N-1,$$
$$\tilde{\vartheta}_{i,0}=0 \quad \text{and} \quad \tilde{\vartheta}_{i,N}=0, \qquad 0\leqslant i\leqslant N,$$
$$\tilde{\vartheta}_{0,j}=\tilde{\vartheta}_{1,j} \quad \text{and} \quad \tilde{\vartheta}_{N,j}=\tilde{\vartheta}_{N-1,j}, \qquad 1\leqslant j\leqslant N-1,$$

where

$$(\delta_1^h \tilde{\vartheta}^h)_{i,j} = \tilde{\vartheta}_{i+1,j} - 2\tilde{\vartheta}_{i,j} + \tilde{\vartheta}_{i-1,j},$$
$$(\delta_2^h \tilde{\vartheta}^h)_{i,j} = \tilde{\vartheta}_{i,j+1} - 2\tilde{\vartheta}_{i,j} + \tilde{\vartheta}_{i,j-1}.$$

4.3. Problem ($\mathscr{P}_4$)

Using the unknown $C = \tilde{C} + s$, we write this elliptical linear problem as

$$\operatorname{div}((-\nu_2+\sigma)\nabla C^{m+1}) = -\operatorname{div}(C^m(\bar{\mathbf{u}}^m + \beta(C^m)) - \sigma\nabla C^m) \quad \text{in } \Omega,$$
$$C^{m+1} = -1 \quad \text{on } \Gamma_1, \qquad C^{m+1} = 1 \quad \text{on } \Gamma_2, \qquad \partial_1 C^{m+1} = 0 \quad \text{on } \Gamma_3 \cup \Gamma_4.$$

To recover the discrete compatibility condition (19) with $b^h = -(k\partial_2 C^{m+1})^h$ (Corollary 1), the following discrete approach must be used:

$$(\mathscr{D}_h((-\nu_2+\sigma^h)(\nabla^h C^{m+1,h})))_{i,j} = -(\mathscr{D}_h(C^{m,h}(\bar{\mathbf{u}}^{m,h} + \beta(C^{m,h})) - \sigma^h \nabla^h C^{m,h}))_{i,j}, \quad 0 \leqslant i,j \leqslant N-1, \tag{20}$$

$$(C^{m+1})_{i,0} = -1 \quad \text{and} \quad (C^{m+1})_{i,N} = 1, \quad 0 \leqslant i \leqslant N-1, \tag{21}$$

$$(\nabla_1^h C^{m+1,h})_{0,j} = 0 \quad \text{and} \quad (\nabla_1^h C^{m+1,h})_{N,j} = 0, \quad 0 \leqslant j \leqslant N-1, \tag{22}$$

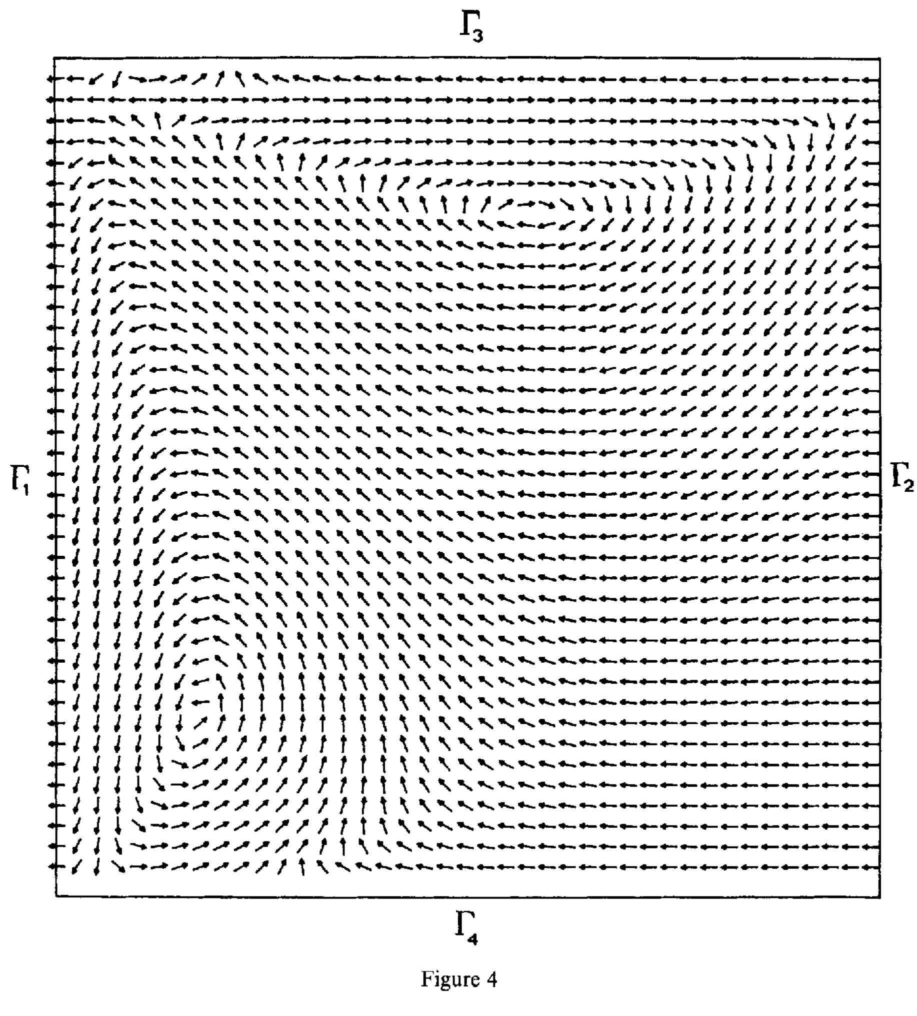

Figure 4

where

$$(\nabla^h \varphi^h)_{i,j} = ((\nabla_1^h \varphi^h)_{i,j}, (\nabla_2^h \varphi^h)_{i,j}) = (1/h)\,(\varphi_{i,j} - \varphi_{i-1,j}, \varphi_{i,j} - \varphi_{i,j-1}),$$
$$(\mathscr{D}_h \mathbf{u}^h)_{i,j} = (1/h)\,[(u_{1_{i+1,j}} - u_{1_{i,j}}) + (u_{2_{i,j+1}} - u_{2_{i,j}})].$$

Proposition 1

The solution $C^{m+1,h}$ of (20)–(22) satisfies

$$k_1 \sum_{i=0}^{N-1} (\nabla_2^h C^{m+1,h})_{i,0} = k_2 \sum_{i=0}^{N-1} (\nabla_2^h C^{m+1,h})_{i,N}. \tag{23}$$

Proof. By summation of equations (20) for $0 \leqslant i, j \leqslant N-1$ and the boundary conditions on Γ we have

$$(v_2 - k_1) \sum_{i=0}^{N-1} (\nabla_2^h C^{m+1,h})_{i,0} = (v_2 + k_2) \sum_{i=0}^{N-1} (\nabla_2^h C^{m+1,h})_{i,N}.$$

Thus

$$k_1 \sum_{i=0}^{N-1} (\nabla_2^h C^{m+1,h})_{i,0} - k_2 \sum_{i=0}^{N-1} (\nabla_2^h C^{m+1,h})_{i,N} = [k_1 - k_2(v_2 - k_1)/(v_2 + k_1)] \sum_{i=0}^{N-1} (\nabla_2^h C^{m+1,h})_{i,0} = 0.$$

Corollary 1

If we make $(k\nabla_2^h C^{m+1,h})_{0,1} = k_1(\nabla_2^h C^{m+1,h})_{0,0}$, then the solution $C^{m+1,h}$ of system (20)–(22) satisfies condition (19) with $b^h = -(k^h \nabla_2^h C^{m+1,h})$.

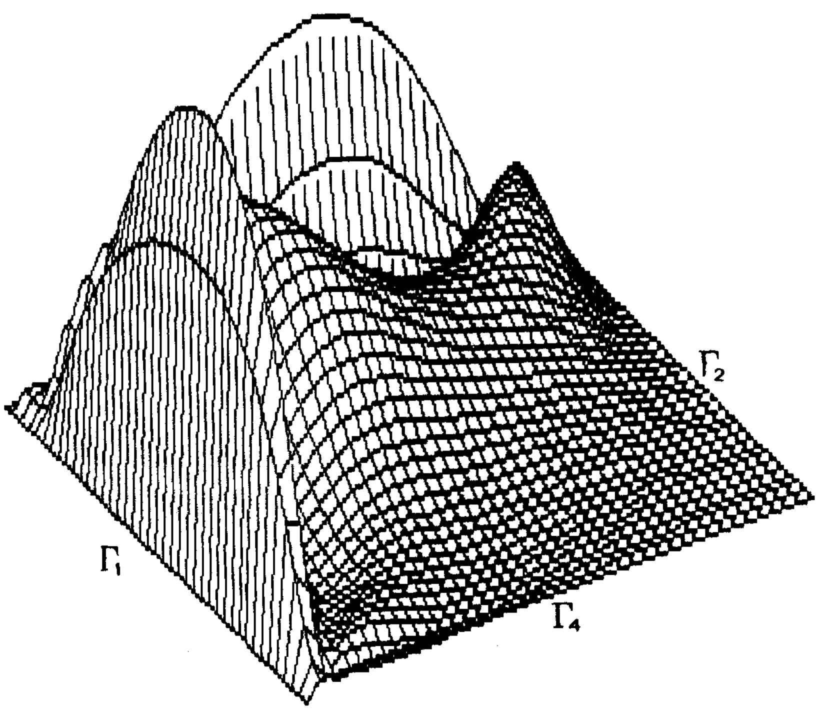

Figure 5

Then we can use the discrete Galerkin approach (18) for solving problems ($\mathscr{P}_1$) and ($\mathscr{P}_2$) and the iterative method is well-posed.

5. NUMERICAL RESULTS

From the numerical experiments performed we have selected three representative cases of the different behaviours observed.

In all cases we have considered $k(x_1, x_2) = k_1 + (k_2 - k_1)x_2$, $L = 1$ and the steps in both directions x_1 and x_2 equal to 1/40. For smaller steps the results are identical—the results are not subject to reduction of step—while for bigger steps details of the behaviour of the solution are progressively lost.

Table I

m	$\frac{\lvert u_1^{mh} - u_1^{m-1h} \rvert_2^2}{\lvert u_1^{m-1h} \rvert_2^2}$	$\frac{\lvert u_2^{mh} - u_2^{m-1h} \rvert_2^2}{\lvert u_2^{m-1h} \rvert_2^2}$
2	$0{\cdot}2381003 \times 10$	$0{\cdot}3916987 \times 10$
3	0.7577428×10^{-1}	$0{\cdot}9251665 \times 10^{-1}$
4	$0{\cdot}8425583 \times 10^{-3}$	$0{\cdot}9197070 \times 10^{-3}$
5	$0{\cdot}4198000 \times 10^{-4}$	$0{\cdot}4506750 \times 10^{-4}$
6	$0{\cdot}3371395 \times 10^{-5}$	$0{\cdot}3597724 \times 10^{-5}$
7	$0{\cdot}1872095 \times 10^{-6}$	$0{\cdot}1305585 \times 10^{-6}$
8	$0{\cdot}1272534 \times 10^{-7}$	$0{\cdot}1066720 \times 10^{-7}$
9	$0{\cdot}9498784 \times 10^{-9}$	$0{\cdot}7291265 \times 10^{-9}$
10	$0{\cdot}7382492 \times 10^{-10}$	$0{\cdot}6080274 \times 10^{-10}$
11	$0{\cdot}5191394 \times 10^{-11}$	$0{\cdot}4255035 \times 10^{-11}$
12	$0{\cdot}4092533 \times 10^{-12}$	$0{\cdot}3143575 \times 10^{-12}$
13	$0{\cdot}3021557 \times 10^{-13}$	$0{\cdot}2640049 \times 10^{-13}$
14	$0{\cdot}2288871 \times 10^{-14}$	$0{\cdot}1713898 \times 10^{-14}$
15	$0{\cdot}1772135 \times 10^{-15}$	$0{\cdot}1538949 \times 10^{-15}$

Table II

m	$\frac{\lvert u_1^{mh} - u_1^{m-1h} \rvert_2^2}{\lvert u_1^{m-1h} \rvert_2^2}$	$\frac{\lvert u_2^{mh} - u_2^{m-1h} \rvert_2^2}{\lvert u_2^{m-1h} \rvert_2^2}$
2	$0{\cdot}1278433 \times 10$	$0{\cdot}1309131 \times 10$
3	$0{\cdot}7299503$	$0{\cdot}1500075 \times 10$
4	$0{\cdot}3349531 \times 10^{-1}$	$0{\cdot}1284974$
5	$0{\cdot}3473205 \times 10^{-1}$	$0{\cdot}4671985 \times 10^{-1}$
6	$0{\cdot}1962569 \times 10^{-1}$	$0{\cdot}2117689 \times 10^{-1}$
7	$0{\cdot}5312811 \times 10^{-2}$	0.4574173×10^{-2}
8	$0{\cdot}2086147 \times 10^{-2}$	$0{\cdot}1505493 \times 10^{-2}$
9	$0{\cdot}1549681 \times 10^{-2}$	$0{\cdot}1125943 \times 10^{-2}$
10	$0{\cdot}8868324 \times 10^{-3}$	$0{\cdot}6939957 \times 10^{-3}$
11	$0{\cdot}4137669 \times 10^{-3}$	$0{\cdot}2740873 \times 10^{-3}$
12	$0{\cdot}2499810 \times 10^{-3}$	$0{\cdot}1750300 \times 10^{-3}$
13	$0{\cdot}1307526 \times 10^{-3}$	$0{\cdot}1024339 \times 10^{-3}$
14	$0{\cdot}7945335 \times 10^{-4}$	$0{\cdot}5245996 \times 10^{-4}$
15	$0{\cdot}4172027 \times 10^{-4}$	$0{\cdot}3304745 \times 10^{-4}$

For the benefit of the reader we show the field directions and modulus of velocity for different values of m (number of iterations) in order to underline the convergence of the iterative process. Finally, we show tables with the relative differences between the velocity components and the sequential iterations.

5.1. *First case*

$k_1=0{\cdot}239\,86$, $k_2=0{\cdot}240\,11$ $(v_2=460{\cdot}742\,27)$, $\mu=415$, $v_1=440$, $\gamma_1=\gamma_2=0{\cdot}5$.

Thus we have

$$\lambda=35{\cdot}982\,363,\quad v=415,\quad \rho=180{\cdot}126\,56,\quad f=365{\cdot}116\,02.$$

Condition (9) is satisfied and the numerical scheme converges rapidly to the solution, as can be seen in Table I. The field of velocity directions of the solution corresponding to iteration 15 is shown in Figure 4 and the modulus of these velocities is shown in Figure 5. The same figures are

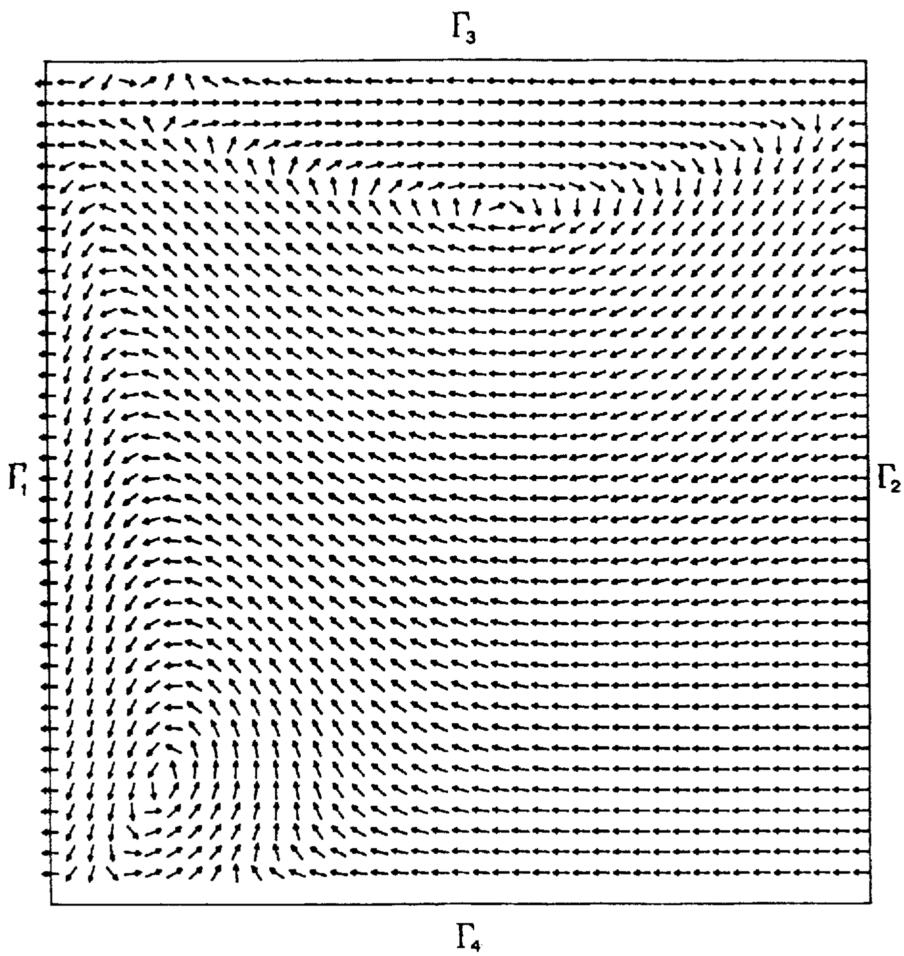

Figure 6

obtained when we draw the field of velocity directions and the modulus of these velocities corresponding to iteration 5.

5.2. Second case

$k_1 = 0{\cdot}4593, \quad k_2 = 0{\cdot}4609 \quad (\nu_2 = 264{\cdot}614\,21), \quad \mu = 249, \quad \nu_1 = 262, \quad \gamma_1 = \gamma_2 = 0.5.$

Thus we have

$$\lambda = 87{\cdot}785\,570, \qquad \nu = 248{\cdot}501\,83, \qquad \rho = 478{\cdot}527\,27, \qquad f = 1172{\cdot}531\,48.$$

Condition (9) is not satisfied, but it is 'close', and the numerical scheme converges.

Of course, we obtain numerical solutions since the theoretical condition of convergence is less restrictive than (9). From Table II we deduce that a slower convergence is obtained.

The field of velocity directions of the solution corresponding to iteration 15 is shown in Figure 6. If the field of velocity directions corresponding to iteration 5 is drawn, the figure obtained is slightly different from Figure 6.

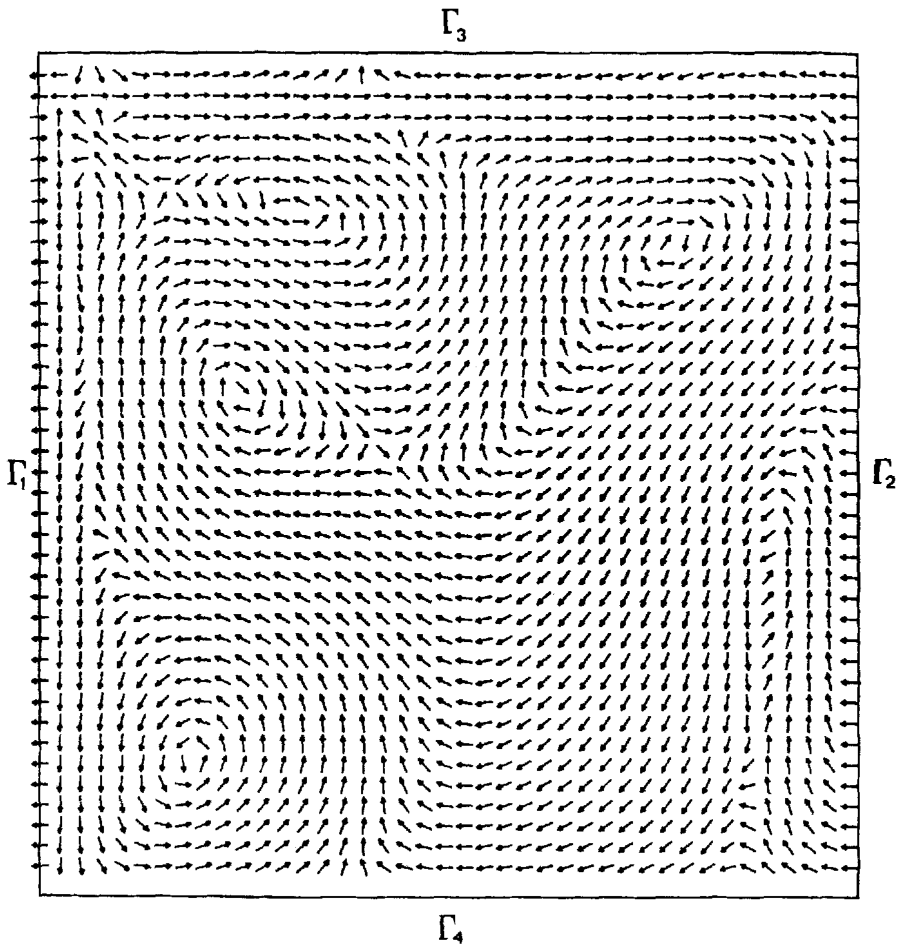

Figure 7

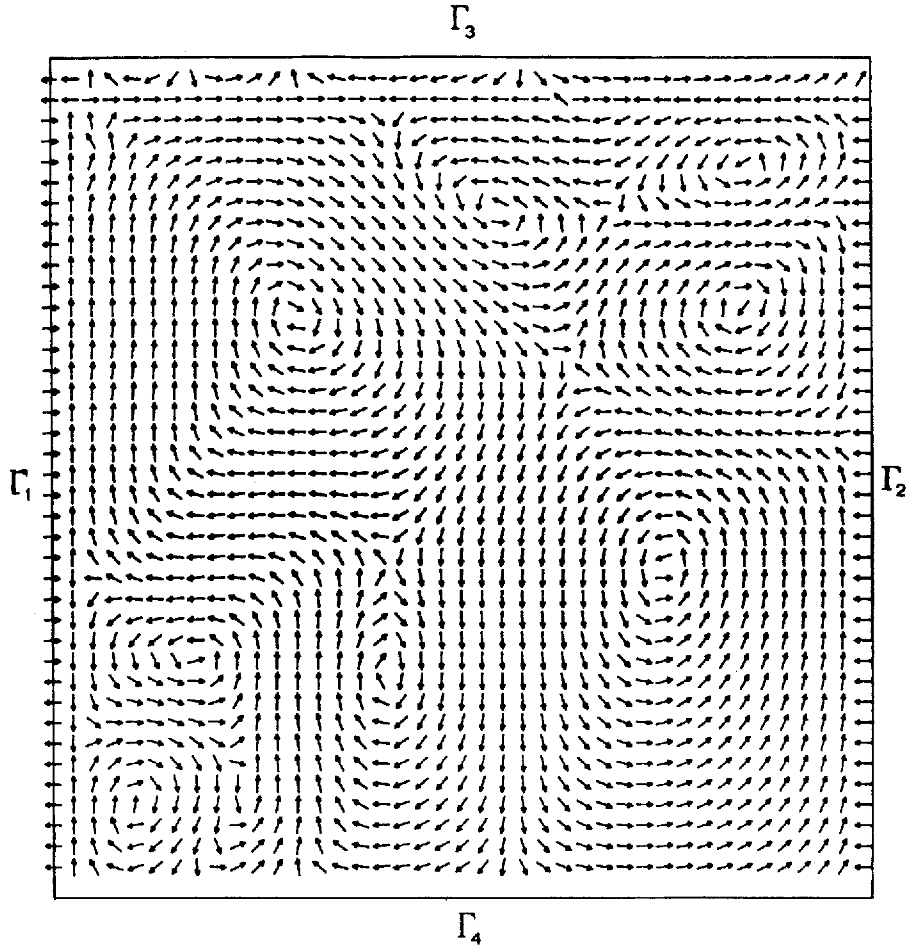

Figure 8

5.3. Third case

$$k_1 = 0{\cdot}95, \quad k_2 = 1 \quad (\nu_2 = 38), \quad \mu = 40, \quad \nu_1 = 40, \quad \gamma_1 = \gamma_2 = 1.$$

Thus we have

$$\lambda = 290{\cdot}300\,88, \quad \nu = 34{\cdot}483\,077, \quad \rho = 1744{\cdot}6245, \quad f = 5046{\cdot}048.$$

Condition (9) is largely not satisfied and the iterative process does not converge.

Figures 7 and 8 represent the fields of velocity directions for iterations 2 and 3 respectively. In this case there is a greater number of vortices that are larger at iteration 3.

At iteration 4, overflow is obtained.

From these results we conclude that a constraint is needed on the physical parameters of the problem in order to achieve the existence of a solution of the steady state problem $(\mathscr{P})$.

ACKNOWLEDGEMENT

M. L. and F. V. were partially supported by Grant UPV 127.310-0170/89 of the Universidad del País Vasco.

REFERENCES

1. A. J. Chorin and J. E. Marsden, *A Mathematical Introduction to Fluid Mechanics*, Springer, New York, 1979.
2. M. Lezáun, 'Un problema de contorno de la mecánica de fluídos', *Tesis*, Universidad del País Vasco, Bilbao, 1986.
3. M. Gaultier and M. Lezáun, 'An existence and uniqueness theorem for the transfer of mass and heat in a rectangular cavity', *IMA J. Appl. Math.*, **48**, 125–148 (1992).
4. R. Teman, *Navier–Stokes Equations: Theory and Numerical Analysis*, North-Holland, Amsterdam, 1979.
5. A. B. Stephens, J. B. Bell, J. M. Solomon and L. B. Hackerman, 'A finite difference Galerkin formulation for the incompressible Navier–Stokes equations', *J. Comput. Phys.*, **53**, 152–172 (1984).
6. B. Friedman, *Principles and Techniques of Applied Mathematics*, Wiley, New York, 1956.
7. A. R. Mitchel and D. F. Griffiths, *The Finite Differnce Method in Partial Differential Equations*, Wiley, New York, 1980.